(Defn) **Eigenvector or Characteristic vector:-** Let T be an operator on a Hilbert Space H. A non-zero vector $x \in H$ is said to be an Eigenvector T, if $Tx = \lambda x$ for some scalar λ.

(Defn) **Eigenvalue:-** If T be an operator on a Hilbert Space H. A scalar λ for which $Tx = \lambda x$ where x be a non-zero vector of H is known as Eigenvalue or Characteristic value of T.

(Defn) **Eigenspace:-** Let λ be an eigenvalue of an operator T on a non-zero Hilbert space H.

Let $M = \{x : x \in H \,\&\, (T - \lambda I)x = 0\}$ i.e. $Tx - \lambda Ix = 0$

i.e. $Tx - \lambda x = 0$ i.e. $Tx = \lambda x$.

M is said to be the ~~eigenvector~~ eigenspace of T corresponding to λ.

Clearly, $M \subseteq H$, Clearly, $0 \in M$ since,

$T0 - 0 = \lambda 0$

Hence M is non-empty. Let $x, y \in M$ be arbitrary and α, β any two scalar

$\therefore Tx = \lambda x$, $Ty = \lambda y$ [$\because T$ is linear]

$T(\alpha x + \beta y) = \alpha Tx + \beta Ty = \alpha\lambda x + \beta\lambda y$

$= \lambda(\alpha x + \beta y)$. $\therefore \alpha x + \beta y \in M$.

Hence, M is linear sub-spaces of H.

Finally $M = (T - \lambda I)^{-1}(0)$. Hence, M is kernel

$T-\lambda I$, therefore, M is closed.

NOTE:- M is said to be invariate under T since $Tx = \lambda x \Rightarrow \lambda x \in M$ if $x \in M$.

V.V. Imp. M.U.

M.Sc. 93, 95

QNo> Define Spectrum. If T is an arbitrary operator on a finite dimensional non-zero Hilbert space H. $\lambda_1, \lambda_2, ---- \lambda_m$ are distinct eigenvalues of T and $M_1, M_2, ---- M_m$ are their corresponding eigen spaces, ~~then~~ and $P_1, P_2, ----- P_m$ are projections on these eigen spaces. Then for the statements.

(i) The M_i's are pairwise orthogonal and span H.

(ii) The P_i's are pairwise orthogonal.

$$I = \sum_{i=1}^{m} P_i \text{ and } T = \sum_{i=1}^{m} \lambda_i P_i.$$

(iii) T is a normal operator.

Prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

M.U.
M.Sc. 95
M.U. 94
S. Paper

Or, QNo> State and Prove Spectral theorem for normal operators on a finite-dimensional Hilbert space.

or QNo> (Spectral theorem): The following statements are equivalent to one another

(i) the eigenspaces M_i's are pairwise orthogonal and span H, i.e.

$$H = M_1 \oplus M_2 \oplus M_3 \oplus ----- \oplus M_m.$$

(ii) The Projection P_i's are Pairwise orthogonal and

$$I = \sum_{i=1}^{m} P_i \quad \text{and} \quad T = \sum_{i=1}^{m} \lambda_i P_i$$

(iii) T is normal.

NOTE:- ऊपर वाला Q No. में सिर्फ Proof देना है, तथा 9b वाला में Statement भी देना है,

Ans:-> (Defn.) Spectrum:- The Set of all eigenvalues of an operator T on H is Called the spectrum of T and is denoted by $\sigma(T)$.

Statement:- The following Statements are equivalent to one another.

(i) The M_i's are Pairwise orthogonal and Span H.

(ii) The P_i's are Pairwise orthogonal,

$$I = \sum_{i=1}^{m} P_i \quad \text{and} \quad T = \sum_{i=1}^{m} \lambda_i P_i$$

(iii) T is a normal operator.

Proof:- (i) $\Rightarrow$ (ii). We assume (i). Then every vector x in H can be expressed uniquely in the form,

$$x = x_1 + x_2 + \text{------} + x_m \quad \text{— (1)}$$

Where, $x_i \in M_i$ and $x_i \perp x_j$ as $i \neq j$.

Therefore, $Tx = T(x_1 + x_2 + \text{----} + x_m)$

$= Tx_1 + Tx_2 + \text{----} + Tx_m$.

Since, $x_i \in M_i$, $Tx_i = \lambda_i x_i \, (i = 1, 2, \text{----} m)$.

$\therefore Tx = \lambda_1 x_1 + \lambda_2 x_2 + \text{----} + \lambda_m x_m$ — (2)

Since, M_i's are Pairwise orthogonal, we have

$M_j \subseteq M_i^{\perp}$, $i \neq j$

~~$P_i x = x_i$~~ Hence from ①, we have

$P_i x = x_i$.

Therefore, $Ix = x = x_1 + x_2 + \cdots + x_m$

$= P_1 x + P_2 x + \cdots + P_m x$

$= (P_1 + P_2 + \cdots + P_m) x \quad \forall x \in$

$\therefore I = P_1 + P_2 + \cdots + P_m = \sum_{i=1}^{m} P_i$.

Hence, from ②, we have

$Tx = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$

$= \lambda_1 P_1 x + \lambda_2 P_2 x + \cdots + \lambda_m P_m x$

$= (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m) x \quad \forall x \in H$.

$\therefore T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m = \sum_{i=1}^{m} \lambda_i P_i$ — ③

Therefore, (i) $\Rightarrow$ (ii).

We shall now, prove that (ii) $\Rightarrow$ (iii)

Let $I = P_1 + P_2 + \cdots + P_m$ & $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$

$\therefore T^* = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)^*$

$= (\lambda_1 P_1)^* + (\lambda_2 P_2)^* + \cdots + (\lambda_m P_m)^*$.

$= \bar{\lambda}_1 P_1^* + \bar{\lambda}_2 P_2^* + \cdots + \bar{\lambda}_m P_m^*$.

$= \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \cdots + \bar{\lambda}_m P_m$.

Now $TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)(\bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \cdots + \bar{\lambda}_m P_m)$

~~$= |\lambda_1|^2 P_1^2 + |\lambda_2|^2 P_2^2 + \cdots + |\lambda_m|^2 P_m^2$~~

$= \lambda_1 \bar{\lambda}_1 P_1^2 + \lambda_2 \bar{\lambda}_2 P_2^2 + \cdots + \lambda_m \bar{\lambda}_m P_m^2$

$= |\lambda_1|^2 P_1^2 + |\lambda_2|^2 P_2^2 + \cdots + |\lambda_m|^2 P_m^2$.

$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m$ [$\because P_i P_j = 0$ for $i \neq j$]

Similarly, $T^* T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m$.

$\therefore TT^* = T^* T$.

Therefore, T is a normal operator.

Thus (ii) $\Rightarrow$ (iii)

Thus, we have proved,

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

~~Rem~~ An arbitrary operator T on H need not have an eigenvalue.

For Consider the operator T on l_2 defined by

$T(x_1, x_2, \cdots) = (0, x_1, x_2 \cdots)$.

Consider the equation $Tx = \lambda x$

i.e. $(0, x_1, x_2 \cdots) = \lambda(x_1, x_2 \cdots)$ — (1)

where $x_2 \neq 0$. For $\lambda = 0$, (1) is not satisfied.

In case $\lambda \neq 0$, then $\lambda x_1 = 0$ and so $x_1 = 0$. Also $\lambda x_2 = x_1 = 0$, $\therefore x_2 = 0$ which is a contradiction.

Thus no scalar λ can satisfy (1) for some non-zero x. Thus the above operator T on l_2 does not have any eigenvalue.

M.Sc. 92, M.Sc. 96, 92

Q.No.→ Let T be a normal operator on a finite dimensional non-zero Hilbert space H with spectrum $\{\lambda_1, \lambda_2, ----, \lambda_m\}$. Using the spectral resolution of T prove the following statements:-

(i) T is self-adjoint $\Leftrightarrow$ each λ_i is real.

(ii) T is positive $\Leftrightarrow$ $\lambda_i \geq 0$ for each i

(iii) T is unitary $\Leftrightarrow$ $|\lambda_i| = 1$ for each i.

M.U. 94, 98, Q.No.→ Let T be a normal operator on a finite-dimensional Hilbert space having the spectrum $\{\lambda_1, \lambda_2, ----, \lambda_m\}$ then prove that

(i) T is self-adjoint $\Leftrightarrow$ each λ_i is real

(ii) T is unitary $\Leftrightarrow$ $|\lambda_i| = 1$ for each i

Solⁿ If $M_1, M_2, ------, M_m$ be the eigenspaces of T corresponding to eigenvalues $\lambda_1, \lambda_2, ----, \lambda_m$ and $P_1, P_2, ----, P_m$ be the projections on $M_1, M_2, -----, M_m$ respectively then the normal operator T has a spectral resolution,

$$T = \lambda_1 P_1 + \lambda_2 P_2 + ---- + \lambda_m P_m.$$

$$\therefore T^* = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + ---- + \bar{\lambda}_m P_m$$

(i) T is self-adjoint $\Leftrightarrow T = T^*$.

$$\Leftrightarrow \lambda_1 P_1 + \lambda_2 P_2 + --- + \lambda_m P_m = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + --- + \bar{\lambda}_m P_m$$

$$\Leftrightarrow (\lambda_1 - \bar{\lambda}_1) P_1 + (\lambda_2 - \bar{\lambda}_2) P_2 + ---- + (\lambda_m - \bar{\lambda}_m) P_m = 0 \quad \text{—(1)}$$

Clearly, if each λ_i is real then $\lambda_i = \overline{\lambda_i}$ for each i. Hence ① is satisfied, therefore, T is self-adjoint.

Now, since P_i's are pairwise orthogonal, if T is self-adjoint then since,

$$(\lambda_1 - \overline{\lambda}_1)P_1 + (\lambda_2 - \overline{\lambda}_2)P_2 + \dots + (\lambda_m - \overline{\lambda}_m)P_m = 0,$$

we have

$$P_i\{(\lambda_1 - \overline{\lambda}_1)P_1 + (\lambda_2 - \overline{\lambda}_2)P_2 + \dots + (\lambda_i - \overline{\lambda}_i)P_i + \dots + (\lambda_m - \overline{\lambda}_m)P_iP_m\}$$

$$= P_i \cdot 0 = 0.$$

$$\therefore (\lambda_1 - \overline{\lambda}_1)P_iP_1 + (\lambda_2 - \overline{\lambda}_2)P_iP_2 + \dots + (\lambda_i - \overline{\lambda}_i)P_i + \dots + (\lambda_m - \overline{\lambda}_m)P_iP_m = 0$$

But, $P_iP_j = 0$ for $i \neq j$. $\therefore P_iP_1 = 0, P_iP_2 = 0, \dots, P_iP_m = 0$.

$\therefore (\lambda_i - \overline{\lambda}_i)P_i = 0$ $\therefore \lambda_i - \overline{\lambda}_i = 0$ or, $\lambda_i = \overline{\lambda}_i$.

Therefore, each λ_i is real. Thus if T is self adjoint, each λ_i is real. i.e. T is self adjoint $\Leftrightarrow$ each λ_i is real.

(ii) For $x \in H$, $(Tx, x) = \sum_{i=1}^{m} \lambda_i \|P_i x\|^2$ —— ②

If T is a +ve operator then $(Tx, x) \geq 0 \ \forall x \in H$.

$\therefore$ from ② $\sum_{i=1}^{m} \lambda_i \|P_i x\|^2 \geq 0, \ \forall x \in H$.

Now, for any fixed i, let $x \in$ range of P_i. then $P_i x = x$ & $P_j x = 0$ for $j \neq i$.

$\therefore \lambda_i \|x\|^2$ $\therefore \lambda_i \geq 0$ for each i.

Conversely, if each $\lambda_i \geq 0$, then each λ_i is real.

Hence by (i) T is self-adjoint.

Also $\|P_i x\|^2 \geq 0$ $\therefore (Tx, x) \geq 0$ $\therefore$ T is positive.

i.e. T is positive $\Leftrightarrow \lambda_i \geq 0$ for each i.

(iii) $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$.

T unitary $\Leftrightarrow TT^* = T^*T = I$.

Now, $TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)^*$.

$= (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)(\bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \cdots + \bar{\lambda}_m P_m)$

$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m$.

If $|\lambda_i| = 1$ for each i, then $TT^* = P_1 + P_2 + \cdots + P_m = I$.

Similarly $T^*T = P_1 + P_2 + \cdots + P_m = I$.

Thus, $TT^* = T^*T = I$. Hence T is unitary.

Conversely, if T is unitary then $TT^* = T^*T = I$.

So, $|\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots |\lambda_m|^2 P_m = I$

$\therefore P_i(|\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_i|^2 P_i^2 + \cdots + |\lambda_m|^2 P_i P_m = P_i$.

$\therefore |\lambda_i|^2 P_i = P_i$ $[\because P_i^2 = P_i]$

$(|\lambda_i|^2 - 1) P_i = 0$ for each i.

Since $P_i \neq 0$, So $|\lambda_i|^2 - 1 = 0$ or, $|\lambda_i| = 1$ for each i.

i.e. T is unitary $\Leftrightarrow |\lambda_i| = 1$ for each i.